

Quantum back-reaction problems*

Ralf Schützhold†

Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany

The macroscopic behavior of many physical systems can be approximately described by classical quantities. However, quantum theory demands the existence of omnipresent quantum fluctuations on top of this classical background – which, albeit small, should have some impact onto its dynamics. The correct treatment of this quantum back-reaction is one of the main problems in quantum gravity and related to fundamental questions such as the initial (big bang) singularity or the cosmological constant. By means of the qualitative analogy between gravity and fluid dynamics, we try to shed some light onto these problems and show some of the difficulties associated with the calculation of the quantum back-reaction starting from the classical (macroscopic) equation of motion.

MOTIVATION

In physics (and other sciences), it is often useful to make *gedanken* experiments in order to explore the limiting cases of a given theory and to approach it from different sides. Therefore, let us do the same here and imagine the following situation: While sitting on the coffee table in your physics department, a colleague approaches you to ask you a favor: “Hi, I have a question for you – there is this theory with the following properties:”

- The classical equations of motion are known and my experimental collaborators have them reasonably well tested at large length scales and low energies.
- However, we all agree that this classical description should break down at small length scales, where somehow quantum effects should become important.
- But if I try to quantize this theory in the usual perturbative manner, I get non-renormalizable UV-divergences and apparently I have to introduce new (unknown) parameters.
- Unfortunately, I do not know and understand the the full quantum theory (yet).

“Nevertheless, my experimental collaborators are now preparing precision measurements and they want me to estimate the back-reaction of the quantum fluctuations onto the classical background profile (as described by the classical equations of motion). Do you have any idea how to do this?”

FLUID DYNAMICS

From reading the above points, one might have expected that this colleague works in quantum gravity – but in fact, precisely the same problem does also apply to fluids dynamics:

- The classical description is provided by the Euler equation

$$\frac{d}{dt}\vec{v} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\vec{\nabla} p}{\varrho} + \frac{\vec{f}_{\text{ext}}}{\varrho}, \quad (1)$$

together with the equation of continuity $\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$ plus the equation of state $p = p(\rho)$ and the external force density \vec{f}_{ext} .

- However, this equation clearly breaks down at small length scales, where the continuum fluid picture fails.

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† E-mail: schuetz@theory.phy.tu-dresden.de

- The perturbative quantization of the longitudinal degrees of freedom yields phonons as quasi-particles and the inclusion of phonon loops produces non-renormalizable UV-divergences. Moreover, the quantization of the transversal degrees of freedom, i.e., the vorticity $\vec{\nabla} \times \vec{v}$, cannot be inferred from the Euler equation at all. E.g., in superfluids, one can only have an integer number of vortices and the circulation quantum of a single vortex depends on the mass m of the constituent particles – which is not contained in Euler equation.
- Even in those situations where the full quantum theory (i.e., the many-particle Hamiltonian) is in principle known, it is practically impossible to solve – except in a few simple cases. (We shall study one of them below, see also [1].)

FIRST TRY

Clearly, the phonon modes derived from Eq. (1) must also possess quantum fluctuations and these quantum fluctuations should somehow react back onto the fluid. Let us try to estimate this impact. In our first attempt, we assume zero knowledge of the microscopic structure of the fluid and just use the classical equation of motion. In order to avoid the aforementioned problems with vorticity, we assume an irrotational flow as described by the Bernoulli equation

$$\frac{\partial}{\partial t}\phi + \frac{1}{2}(\vec{\nabla}\phi)^2 + V_{\text{ext}} = -h(\varrho) = -g\varrho, \quad (2)$$

with ϕ being the velocity potential $\vec{v} = \vec{\nabla}\phi$. The specific enthalpy $dh = dp/\varrho$ determines the equation of state, where we have chosen the relation $h(\varrho) = g\varrho$ describing Bose-Einstein condensates for later comparison.

Now the usual procedure consists in replacing the c-number fields ρ and ϕ in the equation above by operators (more precisely, operator-valued distributions) and to interpret their expectation values as the classical contribution (mean-field expansion)

$$\begin{aligned} \phi &\rightarrow \hat{\phi} = \langle \hat{\phi} \rangle + \delta\hat{\phi} = \phi_{\text{cl}} + \delta\hat{\phi} \\ \varrho &\rightarrow \hat{\varrho} = \langle \hat{\varrho} \rangle + \delta\hat{\varrho} = \varrho_{\text{cl}} + \delta\hat{\varrho}. \end{aligned} \quad (3)$$

Inserting the above split into the Bernoulli equation (2) and taking the expectation value yields

$$\dot{\phi}_{\text{cl}} + \frac{1}{2}(\vec{\nabla}\phi_{\text{cl}})^2 + V_{\text{ext}} + g\varrho_{\text{cl}} = -\frac{1}{2}\langle (\vec{\nabla}\delta\hat{\phi})^2 \rangle, \quad (4)$$

where the last term can be attributed to the quantum fluctuations of the phonon modes and should include their zero-point pressure. The quantum fluctuations $\delta\hat{\phi}$ of sound derived from Eq. (2) behave exactly like a scalar (Klein-Fock-Gordon) quantum field – provided that one replaces the speed of light with the sound velocity c_s in all expressions. Hence the expectation value in the above equation is UV-divergent and behaves as $\int k d^3k$. Remembering that the fluid picture is only valid for large length scales, one would cut off the k -integration at some maximum wavenumber k_{cut} . Below this cut-off k_{cut} , our description should work fairly well and hence one would arrive at the following estimate (see also [2])

$$p_{\text{zero-point}} = -\frac{1}{2}\langle (\vec{\nabla}\delta\hat{\phi})^2 \rangle \propto k_{\text{cut}}^4. \quad (5)$$

An analogous line of reasoning has been adopted for estimates of the cosmological constant (i.e., the gravitational impact of the quantum vacuum) $\Lambda \propto M_{\text{Planck}}^4$, which turns out to be many orders of magnitude too large. However, in the above approach the main contribution to the zero-point pressure stems from phonon modes with wavenumbers close to k_{cut} , where the theory breaks down. Therefore, one might worry whether the above estimate is reliable.

SECOND TRY

In order to tackle this question, let us make a second attempt by incorporating some knowledge about what is going on at high wavenumbers k_{cut} , i.e., how the fluid picture changes at small length scales. Of course, this behavior will depend on the kind of fluid under consideration – i.e., we have to specify it. In the following, we shall consider dilute Bose-Einstein condensates, since we understand the microscopic structure of these quantum fluids quite well.

In Bose-Einstein condensates, the Bernoulli equation breaks down – i.e., acquires additional terms – at the healing length $\xi = 1/\sqrt{mg\varrho}$ ($\hbar = 1$ throughout)

$$\dot{\phi} + \frac{1}{2}(\vec{\nabla}\phi)^2 + V_{\text{ext}} + g\varrho = \frac{1}{2m} \frac{\vec{\nabla}^2 \sqrt{\varrho}}{\sqrt{\varrho}}. \quad (6)$$

For large length scales $k \ll 1/\xi$, the last term[5] can be neglected and we recover the usual sound waves with a linear dispersion relation $\omega = c_s k$. For higher wavenumbers, however, the dispersion relation changes and approaches the energy-momentum relation $\omega = k^2/(2m)$ for free particles (with mass m) in the limit $k\xi \gg 1$

$$\omega^2 = c_s^2 k^2 + \frac{k^4}{4m^2}. \quad (7)$$

If we now insert the same mean-field expansion (3) as in the previous Section into the modified Bernoulli equation (6) and neglect terms of third or higher order in $\delta\hat{\varrho}$, we get

$$p_{\text{zero-point}} = -\frac{1}{2}\langle(\vec{\nabla}\delta\hat{\phi})^2\rangle + \frac{1}{8\varrho^2}\langle(\vec{\nabla}\delta\hat{\varrho})^2\rangle + \mathcal{O}(\delta\hat{\varrho}^3). \quad (8)$$

Without respecting the altered dispersion relation (7), the additional term $\langle(\vec{\nabla}\delta\hat{\varrho})^2\rangle$ would be even stronger UV-divergent $\int k^3 d^3k$ than the first one $\langle(\vec{\nabla}\delta\hat{\phi})^2\rangle$. However, including the full dispersion relation (7), both expectation values behave as $\int k^2 d^3k$ for large k and the leading UV-divergences cancel exactly. Nevertheless, the sub-leading UV-singularities remain and one obtains

$$p_{\text{zero-point}} \propto gk_{\text{cut}}^3. \quad (9)$$

Therefore, the k_{cut}^4 behavior in estimate (5) we just “successfully” calculated in the previous Section is completely canceled by the additional term in Eq. (6), which becomes relevant for wavenumbers $k = \mathcal{O}(1/\xi)$ where the fluid picture breaks down. Furthermore, even after fully incorporating the cut-off $k_{\text{cut}}^\xi \sim 1/\xi$ stemming from the healing length, the result is still UV-divergent (though the singularity is weaker now). Naturally, one would expect that this remaining divergence k_{cut}^3 reflects the omitted details of the microscopic structure of two-particle interaction potential which generates the internal pressure in the condensate, cf. [3].

THIRD TRY – FULL THEORY

Fortunately, for dilute Bose-Einstein condensates, we are able to address this question analytically. The full many-particle Hamiltonian describing the condensate reads

$$\hat{H} = \int d^3r \left[\frac{(\vec{\nabla}\hat{\Psi}^\dagger) \cdot (\vec{\nabla}\hat{\Psi})}{2m} + V_{\text{ext}} \hat{\Psi}^\dagger \hat{\Psi} \right] + \int d^3r d^3r' \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}^\dagger(\vec{r}') V_{\text{int}}(\vec{r} - \vec{r}') \hat{\Psi}(\vec{r}) \hat{\Psi}(\vec{r}'), \quad (10)$$

where $V_{\text{int}}(\vec{r} - \vec{r}')$ denotes the two-particle interaction potential. In the s -wave approximation, the microscopic structure of this potential is omitted and the Hamiltonian simplifies to

$$\hat{H} = \int d^3r \left[\frac{(\vec{\nabla}\hat{\Psi}^\dagger) \cdot (\vec{\nabla}\hat{\Psi})}{2m} + V_{\text{ext}} \hat{\Psi}^\dagger \hat{\Psi} + \frac{g}{2} (\hat{\Psi}^\dagger)^2 \hat{\Psi}^2 \right]. \quad (11)$$

In order to make contact to the results of the previous Sections, we introduce the density $\hat{\varrho} = \hat{\Psi}^\dagger \hat{\Psi}$ and phase $\hat{\phi}$ operators via the quantum Madelung ansatz $\hat{\Psi} = e^{i\hat{\phi}} \sqrt{\hat{\varrho}}$. Note, however, that this split is highly singular due to the commutator $[\hat{\Psi}(\vec{r}), \hat{\Psi}^\dagger(\vec{r}')] = \delta^3(\vec{r} - \vec{r}')$. Therefore, if we try to re-express the above Hamiltonian in terms of $\hat{\varrho}$ and $\hat{\phi}$

$$(\hat{\Psi}^\dagger)^2 \hat{\Psi}^2 = (\hat{\Psi}^\dagger \hat{\Psi})^2 - \hat{\Psi}^\dagger [\hat{\Psi}, \hat{\Psi}^\dagger] \hat{\Psi} = \hat{\varrho}^2 - \delta^3(0), \quad (12)$$

we encounter UV-divergent operator-ordering remnants $\delta^3(0) \propto k_{\text{cut}}^3$. Thus the quantum analogue of the Bernoulli equation

$$\frac{\partial \hat{\phi}}{\partial t} = i[\hat{H}, \hat{\phi}] = \frac{\dot{\phi}}{2m} - V_{\text{ext}} - g\hat{\varrho} + \frac{g}{2}\delta^3(0), \quad (13)$$

contains at least one singular term $g\delta^3(0) \propto gk_{\text{cut}}^3$. Comparison with Eq. (9) shows that the estimate of $p_{\text{zero-point}}$ in the previous Section is completely spoilt by such unphysical operator-ordering remnants. One might argue that a constant (though infinite) pressure shift does not play any role – but the coupling constant g may well depend on an external magnetic field and thereby vary spatially (which would feign a pressure gradient).

Note that the singularity $g\delta^3(0) \propto gk_{\text{cut}}^3$ could be cured by a smooth two-particle interaction potential $V_{\text{int}}(\vec{r} - \vec{r}')$. However, insertion of the quantum Madelung split $\hat{\Psi} = e^{i\hat{\phi}} \sqrt{\hat{\rho}}$ into the kinetic term $(\vec{\nabla}\hat{\Psi}^\dagger) \cdot (\vec{\nabla}\hat{\Psi})$ again yields operator-ordering singularities which are independent of $V_{\text{int}}(\vec{r} - \vec{r}')$ and thus the quantum Bernoulli equation above is still ill-defined. Therefore, in order to calculate the back-reaction of the quantum fluctuations correctly, one should not use the ill-defined phase operator $\hat{\phi}$ but employ the fundamental field operators $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ instead. In analogy to (3), the mean-field split $\hat{\Psi} = \psi_c + \hat{\chi}$ into the condensate order parameter ψ_c plus small quantum fluctuations (phonon modes) $\hat{\chi}$ yields the Gross-Pitaevskii equation

$$i\dot{\psi}_c = \left[-\frac{\vec{\nabla}^2}{2m} + V_{\text{ext}} + g|\psi_c|^2 + 2g\langle\hat{\chi}^\dagger\hat{\chi}\rangle \right] \psi_c + g\langle\hat{\chi}^2\rangle\psi_c^* \quad (14)$$

including the quantum back-reaction terms $\langle\hat{\chi}^\dagger\hat{\chi}\rangle$ and $\langle\hat{\chi}^2\rangle$. The first term $\langle\hat{\chi}^\dagger\hat{\chi}\rangle$ is called the quantum depletion and is finite even in the s -wave approximation $V_{\text{int}}(\vec{r} - \vec{r}') \rightarrow g\delta^3(\vec{r} - \vec{r}')$. The second term $\langle\hat{\chi}^2\rangle$ (known as the anomalous contribution) would be UV-divergent in this approximation $V_{\text{int}}(\vec{r} - \vec{r}') \rightarrow g\delta^3(\vec{r} - \vec{r}')$, but only weakly $\langle\hat{\chi}^2\rangle \propto k_{\text{cut}}$. Taking into account the finite range of the interaction $V_{\text{int}}(\vec{r} - \vec{r}') \neq g\delta^3(\vec{r} - \vec{r}')$, however, it is also finite. Therefore, the description in terms of the fundamental field operators $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ is well-defined and finite – whereas the phase operator $\hat{\phi}$ is ill-defined and generates unphysical singularities.

CONCLUSIONS

The presented *gedanken* experiment is based on a qualitative analogy between (quantum) fluids and (quantum) gravity sketched in the following table. Even though the problem of quantizing gravity is roughly as old as the quest for a full quantum description of fluids, our progress in the latter case is far more explicit (see the various question marks below). This difference can probably be mostly attributed to the vastly different input from the experimental side.

Classical equation of motion:	Euler equation (1)	\leftrightarrow	Einstein equations
Quantized linearizations:	Phonons $\hat{\chi}$	\leftrightarrow	Gravitons (?)
First UV cut-off scale:	Healing length ξ	\leftrightarrow	Planck scale M_{Planck}
Full quantum theory:	Many-body Hamiltonian (10)	\leftrightarrow	Quantum gravity?
Quantum back-reaction:	Zero-point pressure $p_{\text{zero-point}}$	\leftrightarrow	Cosmological constant Λ ?

Now, what can we learn from our *gedanken* experiment, i.e., what are possible lessons for quantum gravity (based on our superior understanding of quantum fluids)?

- Usually, quasi-particles are obtained by quantizing the linearized fluctuations of the classical equation of motion. For phonons, this procedure indeed yields the correct description for low energies (on the linear level). However, the vorticity cannot be quantized in this way and requires some knowledge of the microscopic structure of the fluid – which is not contained in the Euler equation. Whether this linearized quantization procedure works for gravitons is not completely clear. One might expect that they behave like Goldstone modes and thus are similar to phonons – but without knowing the full quantum theory, this is not evident.
- Going beyond linear order, one encounters (non-renormalizable) UV divergences in both cases. As we have observed in the previous Sections, naively summing the zero-point fluctuations of the phonon modes up to some cut-off (and thereby extrapolating the Euler equation to large k) does not give the correct result. This observation provides a different view on the naive estimate $\Lambda \propto M_{\text{Planck}}^4$ of the cosmological constant, cf. [4].
- A third point is the existence of two (or more) vastly different UV cut-off scales. In Bose-Einstein condensates, the first deviation from the Bernoulli equation occurs for length scales at the healing length $k_{\text{cut}}^\xi \sim 1/\xi$, where the dispersion relation changes. In gravity, a change of the dispersion relation corresponds to the breakdown or modification of Lorentz invariance. However, even including the modified dispersion relation (7), there are still UV divergences in the s -wave scattering approximation (11), which can be removed by taking into account the

full two-particle interaction potential $V_{\text{int}}(\vec{r} - \vec{r}')$. Therefore, the range of $V_{\text{int}}(\vec{r} - \vec{r}')$ introduces another UV cut-off k_{cut}^g which is (in dilute Bose-Einstein condensates) much larger $k_{\text{cut}}^g \gg k_{\text{cut}}^\xi \sim 1/\xi$ than the wavenumbers at which the effective Lorentz invariance breaks down. This observation poses the question of the significance of the UV scales in quantum gravity – does really everything happen at the Planck scale and is there indeed nothing going on at higher scales?

- In dilute Bose-Einstein condensates with a pure contact interaction, the first cut-off $k_{\text{cut}}^\xi \sim 1/\xi = \sqrt{mg\rho}$ is directly related to the circulation quantum

$$\Gamma = \oint d\vec{r} \cdot \vec{v} = \frac{2\pi}{m} \mathbb{N} = 2\pi\xi c_s \mathbb{N}, \quad (15)$$

which determines the “quantization” of vorticity. In other superfluids, however, the two scales can be independent. In the presence of finite-range (e.g., dipolar) interactions, for example, the first deviation of the dispersion relation is determined by the maxon hill/roton dip, which depends on the interaction structure – while the circulation quantum is still determined by the mass m . (I.e., in this case, there are *three* UV scales.)

- Finally, the presented calculations clearly demonstrate the importance of identifying the correct fundamental variables. Given a classical equation valid at large scales, one can generate almost every outcome for the impact of the quantum fluctuations by quantizing it in terms of different variables and using the operator ordering ambiguities. In fact, the power of fluid dynamics partly relies on the fact that many systems which are totally different on the microscopic level obey very similar classical equation of motions at large length scales and low energies. Therefore, it is crucial to identify universal (emergent) features and to distinguish them from system specific properties.

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 - [5] For historical reasons, this term is called “quantum pressure”, but that nomenclature is a bit misleading since this term already occurs on the classical level – e.g., in Eq. (6), which is purely classical.